# Fourier Analysis 03-23

## Chap 5. The Fourier transform on $\mathbb{R}$.

- A reasonal perodic function on $\mathbb{R}$ can be represented by its Fourier series.

  For instance, for a 1-period function $f$

$$f(x) \sim \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

where $\hat{f}(n) = \int_0^1 f(x) e^{-2\pi i n x} dx$.

- Do we have an analogue for non-periodic functions on $\mathbb{R}$?

§ 5.1 Functions of moderate decrease and integration.

Def. A function $f : \mathbb{R} \to \mathbb{C}$ is said to be of moderate decrease if ① $f$ is cts on $\mathbb{R}$.

② $\exists$ a constant $A > 0$ such that

$$|f(x)| \leq \frac{A}{1+x^2}, \quad \forall x \in \mathbb{R}.$$

Let $\mathcal{M}(\mathbb{R})$ be the collection of all functions on $\mathbb{R}$ of moderate decrease.

**Fact:** $\mathcal{M}(\mathbb{R})$ is a vector space on $\mathbb{C}$.

$\forall f, g \in \mathcal{M}(\mathbb{R})$, $\forall \alpha, \beta \in \mathbb{C}$, we have

$$\alpha f + \beta g \in \mathcal{M}(\mathbb{R}).$$

**Def** (improper integration) Let $f \in \mathcal{M}(\mathbb{R})$. We define

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{N\to\infty} \int_{-N}^{N} f(x)\,dx.$$

Riemann integral.

**Lemma 1.** The above limit always exists.

**Proof.** Let $I_N = \int_{-N}^{N} f(x)\,dx$ for $N \in \mathbb{N}$.

We prove that $(I_N)$ is a Cauchy sequence.

Let $M > N$.

$$|I_M - I_N| = \left| \int_{|x|\le M} f(x)\,dx - \int_{|x|\le N} f(x)\,dx \right|$$

$$= \left| \int_{N\le |x| \le M} f(x)\,dx \right|$$

$$\le \int_{N\le |x|\le M} |f(x)|\,dx$$

$$\le \int_{|x|\geq N} \frac{A}{x^2}\, dx$$

$$= \frac{2A}{N} \to 0 \quad \text{as } N \to \infty$$

Hence $(I_N)$ is a Cauchy sequence. $\blacksquare$.

Lemma 2. Write $L(f) = \int_{-\infty}^{\infty} f(x)\,dx$ for $f \in M(\mathbb{R})$.

Then ① $L$ is linear, i.e. $L(\alpha f + \beta g) = \alpha L(f) + \beta L(g)$ where $f, g \in M(\mathbb{R})$, $\alpha, \beta \in \mathbb{C}$.

② Translation invariance,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} f(x+h)\,dx, \quad \forall h \in \mathbb{R}.$$

③ Scaling under dilation: $\forall\, \delta > 0$,

$$\delta \int_{-\infty}^{\infty} f(\delta x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx.$$

④ (Absolute continuity)

$$\lim_{h\to 0} \int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx = 0$$

Pf. Here we only prove ④

Let $\varepsilon > 0$. Assume $h \in (-1, 1)$.

Take a large $N$ such that

$$\int_{|x|\geq N} |f(x)|\,dx < \varepsilon, \qquad \int_{|x|\geq N} |f(x+h)|\,dx < \varepsilon$$

$$(\ \forall h \in (-1,1)\ )$$

Now $\displaystyle\int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx$

$$\leq \int_{|x|\geq N} |f(x)|\,dx + \int_{|x|\geq N} |f(x+h)|\,dx$$

$$+ \int_{|x|\leq N} |f(x+h) - f(x)|\,dx$$

$$\leq 2\varepsilon + \int_{-N}^{N} |f(x+h) - f(x)|\,dx.$$

(Since $f$ is $\overset{\text{unif}}{\text{cts}}$ on $[-N-1, N+1]$, $\exists\, \delta > 0$ s.t

if $|h| < \delta$, then $|f(x+h) - f(x)| \leq \frac{\varepsilon}{2N}$

$\forall\ x \in [-N, N]$.)

$$\leq 2\varepsilon + \frac{\varepsilon}{2N}\cdot 2N \leq 3\varepsilon \quad \text{when } |h| < \delta.$$

$\boxed{\phantom{x}}$.

## §1.2 The Fourier transform on $\mathbb{R}$.

**Def.** Let $f \in M(\mathbb{R})$. The Fourier transform of $f$ is

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi \cdot x}\, dx, \quad \xi \in \mathbb{R}$$

$$\left( \left| e^{-2\pi i \xi x} \right| = 1, \quad \text{hence} \quad f(x) \cdot e^{-2\pi i \xi x} \in M(\mathbb{R}) \right)$$

- $\hat{f} : \mathbb{R} \to \mathbb{C}$
- $|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)|\, dx < \infty$.

**Example 1.** Let $f(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$

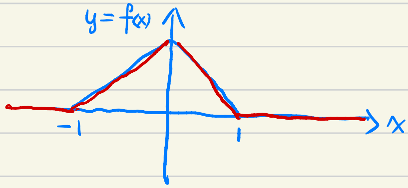
$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-1}^{1} e^{-2\pi i \xi x}\, dx$$

$$= \left. \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \right|_{-1}^{1} = \frac{e^{-2\pi i \xi} - e^{2\pi i \xi}}{-2\pi i \xi}$$

$$= \begin{cases} \dfrac{\sin 2\pi\xi}{\pi\xi} & \text{if } \xi \neq 2 \\ 2 & \text{if } \xi = 0 \end{cases}$$

Example 2. Let $f(x) = \begin{cases} 1 - |x|, & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$




Then $f \in \mathcal{M}(\mathbb{R})$.

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-1}^{1} (1 - |x|)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{0}^{1} (1-x)\, e^{-2\pi i \xi x}\, dx + \int_{-1}^{0} (1+x)\, e^{-2\pi i \xi x}\, dx$$

$$= (1-x) \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \Big|_0^1 - \int_0^1 \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \cdot (-1)\, dx$$

$$(1+x) \frac{e^{-2\pi i \xi x}}{-2\pi i \xi} \Big|_0^1 - \int_0^1 \frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\, dx$$

$$= \begin{cases} \dfrac{\sin^2 \pi \xi}{(\pi \xi)^2} & \text{if} \quad \xi \neq 0 \\ 1 & \text{if} \quad \xi = 0 . \end{cases}$$